What Is Claimed Is:

1   1. A method for using a computer system to solve a global inequality

2 constrained optimization problem specified by a function f and a set of inequality

3 constraints $p_i(\mathbf{x}) \leq 0$ $(i=1,...,m)$, wherein f and p_i are scalar functions of a vector

4 $\mathbf{x} = (x_1, x_2, x_3, ... x_n)$, the method comprising:

5   receiving a representation of the function f and the set of inequality

6 constraints at the computer system;

7   storing the representation in a memory within the computer system;

8   performing an interval inequality constrained global optimization process

9 to compute guaranteed bounds on a globally minimum value of the function $f(\mathbf{x})$

10 subject to the set of inequality constraints;

11   wherein performing the interval inequality constrained global optimization

12 process involves,

13   applying term consistency to a set of relations associated

14   with the global inequality constrained optimization problem over a

15   subbox $\mathbf{X}$, and excluding any portion of the subbox $\mathbf{X}$ that violates

16   any of these relations,

17   applying box consistency to the set of relations associated

18   with the global inequality constrained optimization problem over

19   the subbox $\mathbf{X}$, and excluding any portion of the subbox $\mathbf{X}$ that

20   violates any of these relations, and

21   performing an interval Newton step for the global

22   inequality constrained optimization problem over the subbox $\mathbf{X}$ to

23   produce a resulting subbox $\mathbf{Y}$, wherein the point of expansion of

24   the interval Newton step is a point $\mathbf{x}$.

1   2.   The method of claim 1, wherein applying term consistency to the

2   set of relations involves applying term consistency to the set of inequality

3   constraints $p_i(\mathbf{x}) \leq 0$ ($i=1,...,m$) over the subbox $\mathbf{X}$.

1   3.   The method of claim 1, wherein applying box consistency to the

2   set of relations involves applying box consistency to the set of inequality

3   constraints $p_i(\mathbf{x}) \leq 0$ ($i=1,...,m$) over the subbox $\mathbf{X}$.

1   4.   The method of claim 1,

2   wherein performing the interval inequality constrained global optimization

3   process involves,

4   keeping track of a smallest upper bound f_bar of the

5   function $f(\mathbf{x})$ at a feasible point $\mathbf{x}$,

6   removing from consideration any subbox $\mathbf{X}$ for which

7        $f(\mathbf{X}) > f\_bar$;

8   wherein applying term consistency to the set of relations involves applying

9   term consistency to the f_bar inequality $f(\mathbf{x}) \leq f\_bar$ over the subbox $\mathbf{X}$.

1   5.   The method of claim 4, wherein applying box consistency to the

2   set of relations involves applying box consistency to the f_bar inequality

3    $f(\mathbf{x}) \leq f\_bar$ over the subbox $\mathbf{X}$.

1   6.   The method of claim 1, wherein if the subbox $\mathbf{X}$ is strictly feasible

2   ($p_i(\mathbf{X}) < 0$ for all $i=1,...,n$), performing the interval inequality constrained global

3   optimization process involves:

4   determining a gradient $\mathbf{g}(\mathbf{x})$ of the function $f(\mathbf{x})$, wherein $\mathbf{g}(\mathbf{x})$ includes

5   components $g_i(\mathbf{x})$ ($i=1,...,n$);

29

1   removing from consideration any subbox for which $g(x)$ is bounded away

2 from zero, thereby indicating that the subbox does not include an extremum of

3 $f(x)$; and

4   wherein applying term consistency to the set of relations involves applying

5 term consistency to each component $g_i(x)=0$ ($i=1,...,n$) of $g(x)=0$ over the subbox

6 X.

1   7.   The method of claim 6, wherein applying box consistency to the

2 set of relations involves applying box consistency to each component

3 $g_i(x)=0$ ($i=1,...,n$) of $g(x)=0$ over the subbox X.

1   8.   The method of claim 1, wherein if the subbox X is strictly feasible

2 ($p_i(X) < 0$ for all $i=1,...,n$), performing the interval inequality constrained global

3 optimization process involves:

4   determining diagonal elements $H_{ii}(x)$ ($i=1,...,n$) of the Hessian of the

5 function $f(x)$;

6   removing from consideration any subbox for which a diagonal element

7 $H_{ii}(X)$ of the Hessian over the subbox X is always negative, indicating that the

8 function f is not convex over the subbox X and consequently does not contain a

9 global minimum within the subbox X; and

10   wherein applying term consistency to the set of relations involves applying

11 term consistency to each inequality $H_{ii}(x) \geq 0$ ($i=1,...,n$) over the subbox X.

1   9.   The method of claim 8, wherein applying box consistency to the

2 set of relations involves applying box consistency to each inequality

3 $H_{ii}(x) \geq 0$ ($i=1,...,n$) over the subbox X.

30

1   10.   The method of claim 1, wherein if the subbox **X** is strictly feasible

2   ($p_i(\mathbf{X}) < 0$ for all $i=1,...,n$), performing the interval Newton step involves:

3   computing the Jacobian $\mathbf{J(x,X)}$ of the gradient of the function f evaluated

4   with respect to a point **x** over the subbox **X**; and

5   computing an approximate inverse **B** of the center of $\mathbf{J(x,X)}$,

6   using the approximate inverse **B** to analytically determine the system $\mathbf{Bg(x)}$,

7   wherein $\mathbf{g(x)}$ is the gradient of the function $f(\mathbf{x})$, and wherein $\mathbf{g(x)}$ includes

8   components $g_i(\mathbf{x})$ ($i=1,...,n$).

1   11.   The method of claim 10, wherein applying term consistency to the

2   set of relations involves applying term consistency to each component

3    $(\mathbf{Bg(x)})_i = 0$ ($i=1,...,n$) to solve for the variable x_i over the subbox **X**.

1   12.   The method of claim 10, wherein applying box consistency to the

2   set of relations involves applying box consistency to each component

3    $(\mathbf{Bg(x)})_i = 0$ ($i=1,...,n$) to solve for the variable x_i over the subbox **X**.

1   13.   The method of claim 1, wherein performing the interval Newton

2   step involves performing the Newton step on the John conditions.

1   14.   The method of claim 1,

2   wherein performing the interval inequality constrained global optimization

3   process involves,

4   linearizing the set of inequality constraints to produce a set

5   of linear inequality constraints with interval coefficients that

6   enclose the nonlinear inequality constraints, and

31

1   preconditioning the set of linear inequality constraints

2   through additive linear combinations to produce a set of

3   preconditioned linear inequality constraints; and

4   wherein applying term consistency to the set of relations involves applying

5   term consistency to the set of preconditioned linear inequality constraints over the

6   subbox $\mathbf{X}$.

1   15.   The method of claim 14, wherein applying box consistency to the

2   set of relations involves applying box consistency to the set of preconditioned

3   linear inequality constraints over the subbox $\mathbf{X}$.

1   16.   The method of claim 1, wherein applying term consistency

2   involves:

3   symbolically manipulating an equation within the computer system to

4   solve for a term, $g(x'_j)$, thereby producing a modified equation $g(x'_j) = h(x)$,

5   wherein the term $g(x'_j)$ can be analytically inverted to produce an inverse function

6    $g^{-1}(\mathbf{y})$;

7   substituting the subbox $\mathbf{X}$ into the modified equation to produce the

8   equation $g(X'_j) = h(\mathbf{X})$;

9   solving for $X'_j = g^{-1}(h(\mathbf{X}))$; and

10   intersecting X'_j with the j-th element of the subbox $\mathbf{X}$ to produce a new

11   subbox $\mathbf{X}^+$;

12   wherein the new subbox $\mathbf{X}^+$ contains all solutions of the equation within

13   the subbox $\mathbf{X}$, and wherein the size of the new subbox $\mathbf{X}^+$ is less than or equal to

14   the size of the subbox $\mathbf{X}$.

32

1   17.   The method of claim 1, wherein performing the interval Newton

2   step involves:

3   computing $J(x,X)$, wherein $J(x,X)$ is the Jacobian of the function **f**

4   evaluated as a function of **x** over the subbox **X**; and

5   determining if $J(x,X)$ is regular as a byproduct of solving for the subbox **Y**

6   that contains values of **y** that satisfy $M(x,X)(y-x) = r(x)$, where

7      $M(x,X) = BJ(x,X)$, $r(x) = -Bf(x)$, and **B** is an approximate inverse of the center of

8      $J(x,X)$.

1   18.   A computer-readable storage medium storing instructions that

2   when executed by a computer cause the computer to perform a method for using a

3   computer system to solve a global inequality constrained optimization problem

4   specified by a function f and a set of inequality constraints $p_i(x) \leq 0$ $(i=1,...,m)$,

5   wherein f and p_i are scalar functions of a vector $x = (x_1, x_2, x_3, ... x_n)$, the method

6   comprising:

7   receiving a representation of the function f and the set of inequality

8   constraints at the computer system;

9   storing the representation in a memory within the computer system;

10   performing an interval inequality constrained global optimization process

11   to compute guaranteed bounds on a globally minimum value of the function $f(x)$

12   subject to the set of inequality constraints;

13   wherein performing the interval inequality constrained global optimization

14   process involves,

15   applying term consistency to a set of relations associated

16   with the global inequality constrained optimization problem over a

17   subbox **X**, and excluding any portion of the subbox **X** that violates

18   any of these relations,

33

19   applying box consistency to the set of relations associated

20   with the global inequality constrained optimization problem over

21   the subbox **X**, and excluding any portion of the subbox **X** that

22   violates any of these relations, and

23   performing an interval Newton step for the global

24   inequality constrained optimization problem over the subbox **X** to

25   produce a resulting subbox **Y**, wherein the point of expansion of

26   the interval Newton step is a point **x**.

1   19.   The computer-readable storage medium of claim 18, wherein

2 applying term consistency to the set of relations involves applying term

3 consistency to the set of inequality constraints $p_i(\mathbf{x}) \le 0$ $(i=1,...,m)$ over the

4 subbox **X**.

1   20.   The computer-readable storage medium of claim 18, wherein

2 applying box consistency to the set of relations involves applying box consistency

3 to the set of inequality constraints $p_i(\mathbf{x}) \le 0$ $(i=1,...,m)$ over the subbox **X**.

1   21.   The computer-readable storage medium of claim 18,

2   wherein performing the interval inequality constrained global optimization

3 process involves,

4   keeping track of a smallest upper bound *f_bar* of the

5   function *f*(**x**) at a feasible point **x**,

6   removing from consideration any subbox **X** for which

7   *f*(**X**) > *f_bar*;

8   wherein applying term consistency to the set of relations involves applying

9 term consistency to the *f_bar* inequality *f*(**x**) ≤ *f_bar* over the subbox **X**.

34

1   22.   The computer-readable storage medium of claim 21, wherein

2   applying box consistency to the set of relations involves applying box consistency

3   to the f_bar inequality $f(\mathbf{x}) \le f\_bar$ over the subbox $\mathbf{X}$.

1   23.   The computer-readable storage medium of claim 22, wherein if the

2   subbox $\mathbf{X}$ is strictly feasible $(p_i(\mathbf{X}) < 0$ for all $i=1,...,n)$, performing the interval

3   inequality constrained global optimization process involves:

4   determining a gradient $\mathbf{g}(\mathbf{x})$ of the function $f(\mathbf{x})$, wherein $\mathbf{g}(\mathbf{x})$ includes

5   components $g_i(\mathbf{x})$ $(i=1,...,n)$;

6   removing from consideration any subbox for which $\mathbf{g}(\mathbf{x})$ is bounded away

7   from zero, thereby indicating that the subbox does not include an extremum of

8   $f(\mathbf{x})$; and

9   wherein applying term consistency to the set of relations involves applying

10   term consistency to each component $g_i(\mathbf{x})=0$ $(i=1,...,n)$ of $\mathbf{g}(\mathbf{x})=\mathbf{0}$ over the subbox

11   $\mathbf{X}$.

1   24.   The computer-readable storage medium of claim 23, wherein

2   applying box consistency to the set of relations involves applying box consistency

3   to each component $g_i(\mathbf{x})=0$ $(i=1,...,n)$ of $\mathbf{g}(\mathbf{x})=\mathbf{0}$ over the subbox $\mathbf{X}$.

1   25.   The computer-readable storage medium of claim 18, wherein if the

2   subbox $\mathbf{X}$ is strictly feasible $(p_i(\mathbf{X}) < 0$ for all $i=1,...,n)$, performing the interval

3   inequality constrained global optimization process involves:

4   determining diagonal elements $H_{ii}(\mathbf{x})$ $(i=1,...,n)$ of the Hessian of the

5   function $f(\mathbf{x})$;

35

6   removing from consideration any subbox for which a diagonal element

7        $H_{ii}(\mathbf{X})$ of the Hessian over the subbox $\mathbf{X}$ is always negative, indicating that the

8   function f is not convex over the subbox $\mathbf{X}$ and consequently does not contain a

9   global minimum within the subbox $\mathbf{X}$; and

10   wherein applying term consistency to the set of relations involves applying

11   term consistency to each inequality $H_{ii}(\mathbf{x}) \geq 0$ ($i=1,...,n$) over the subbox $\mathbf{X}$.

1   26.   The computer-readable storage medium of claim 25, wherein

2   applying box consistency to the set of relations involves applying box consistency

3   to each inequality $H_{ii}(\mathbf{x}) \geq 0$ ($i=1,...,n$) over the subbox $\mathbf{X}$.

1   27.   The computer-readable storage medium of claim 18, wherein if the

2   subbox $\mathbf{X}$ is strictly feasible ($p_i(\mathbf{X}) < 0$ for all $i=1,...,n$), performing the interval

3   Newton step involves:

4   computing the Jacobian $\mathbf{J}(\mathbf{x},\mathbf{X})$ of the gradient of the function f evaluated

5   with respect to a point $\mathbf{x}$ over the subbox $\mathbf{X}$; and

6   computing an approximate inverse $\mathbf{B}$ of the center of $\mathbf{J}(\mathbf{x},\mathbf{X})$,

7   using the approximate inverse $\mathbf{B}$ to analytically determine the system $\mathbf{Bg}(\mathbf{x})$,

8   wherein $\mathbf{g}(\mathbf{x})$ is the gradient of the function $f(\mathbf{x})$, and wherein $\mathbf{g}(\mathbf{x})$ includes

9   components $g_i(\mathbf{x})$ ($i=1,...,n$).

1   28.   The computer-readable storage medium of claim 27, wherein

2   applying term consistency to the set of relations involves applying term

3   consistency to each component $(\mathbf{Bg}(\mathbf{x}))_i = 0$ ($i=1,...,n$) to solve for the variable x_i

4   over the subbox $\mathbf{X}$.

1   29.   The computer-readable storage medium of claim 27, wherein
2   applying box consistency to the set of relations involves applying box consistency
3   to each component $(\mathbf{Bg(x)})_i = 0$ ($i=1,...,n$) to solve for the variable x_i over the
4   subbox $\mathbf{X}$.

1   30.   The computer-readable storage medium of claim 18, wherein
2   performing the interval Newton step involves performing the interval Newton step
3   on the John conditions.

1   31.   The computer-readable storage medium of claim 18,
2   wherein performing the interval inequality constrained global optimization
3   process involves,
4   linearizing the set of inequality constraints to produce a set
5   of linear inequality constraints with interval coefficients that
6   enclose the nonlinear inequality constraints, and
7   preconditioning the set of linear inequality constraints
8   through additive linear combinations to produce a set of
9   preconditioned linear inequality constraints; and
10   wherein applying term consistency to the set of relations involves applying
11   term consistency to the set of preconditioned linear inequality constraints over the
12   subbox $\mathbf{X}$.

1   32.   The computer-readable storage medium of claim 31, wherein
2   applying box consistency to the set of relations involves applying box consistency
3   to the set of preconditioned linear inequality constraints over the subbox $\mathbf{X}$.

1   33.   The computer-readable storage medium of claim 18, wherein
2   applying term consistency involves:
3   symbolically manipulating an equation within the computer system to
4   solve for a term, $g(x'_j)$, thereby producing a modified equation $g(x'_j) = h(\mathbf{x})$,
5   wherein the term $g(x'_j)$ can be analytically inverted to produce an inverse function
6   $g^{-1}(\mathbf{y})$;
7   substituting the subbox $\mathbf{X}$ into the modified equation to produce the
8   equation $g(X'_j) = h(\mathbf{X})$;
9   solving for $X'_j = g^{-1}(h(\mathbf{X}))$; and
10   intersecting X'_j with the j-th element of the subbox $\mathbf{X}$ to produce a new
11 subbox $\mathbf{X}^+$;
12   wherein the new subbox $\mathbf{X}^+$ contains all solutions of the equation within
13 the subbox $\mathbf{X}$, and wherein the size of the new subbox $\mathbf{X}^+$ is less than or equal to
14 the size of the subbox $\mathbf{X}$.

1   34.   The computer-readable storage medium of claim 18, wherein
2   performing the interval Newton step involves:
3   computing $\mathbf{J(x,X)}$, wherein $\mathbf{J(x,X)}$ is the Jacobian of the function $\mathbf{f}$
4   evaluated as a function of $\mathbf{x}$ over the subbox $\mathbf{X}$; and
5   determining if $\mathbf{J(x,X)}$ is regular as a byproduct of solving for the subbox $\mathbf{Y}$
6   that contains values of $\mathbf{y}$ that satisfy $\mathbf{M(x,X)(y-x) = r(x)}$, where
7   $\mathbf{M(x,X) = BJ(x,X)}$, $\mathbf{r(x) = -Bf(x)}$, and $\mathbf{B}$ is an approximate inverse of the center of
8   $\mathbf{J(x,X)}$.

1   35.   An apparatus that solves a global inequality constrained
2   optimization problem specified by a function f and a set of inequality constraints

38

3    $p_i(\mathbf{x}) \leq 0$ $(i=1,...,m)$, wherein f and p_i are scalar functions of a vector

4    $\mathbf{x} = (x_1, x_2, x_3, ... x_n)$, the apparatus comprising:

5   a receiving mechanism that is configured to receive a representation of the

6   function f and the set of inequality constraints at the computer system;

7   a memory for storing the representation;

8   an interval global optimization mechanism that is configured to perform

9   an interval inequality constrained global optimization process to compute

10   guaranteed bounds on a globally minimum value of the function $f(\mathbf{x})$ subject to the

11   set of inequality constraints;

12   a term consistency mechanism within the interval global optimization

13   mechanism that is configured to apply term consistency to a set of relations

14   associated with the global inequality constrained optimization problem over a

15   subbox $\mathbf{X}$, and to exclude any portion of the subbox $\mathbf{X}$ that violates any of these

16   relations,

17   a box consistency mechanism within the interval global optimization

18   mechanism that is configured to apply box consistency to the set of relations

19   associated with the global inequality constrained optimization problem over the

20   subbox $\mathbf{X}$, and to exclude any portion of the subbox $\mathbf{X}$ that violates any of these

21   relations, and

22   an interval Newton mechanism within the interval global optimization

23   mechanism that is configured to perform an interval Newton step for the global

24   inequality constrained optimization problem over the subbox $\mathbf{X}$ to produce a

25   resulting subbox $\mathbf{Y}$, wherein the point of expansion of the interval Newton step is

26   a point $\mathbf{x}$.

1   36.   The apparatus of claim 35, wherein the term consistency

2 mechanism is configured to apply term consistency to the set of inequality

3 constraints $p_i(\mathbf{x}) \leq 0$ ($i=1,...,m$) over the subbox $\mathbf{X}$.

1   37.   The apparatus of claim 35, wherein the box consistency

2 mechanism is configured to apply box consistency to the set of inequality

3 constraints $p_i(\mathbf{x}) \leq 0$ ($i=1,...,m$) over the subbox $\mathbf{X}$.

1   38.   The apparatus of claim 35,

2   wherein the interval global optimization mechanism is configured to,

3   keep track of a smallest upper bound f_bar of the function

4          $f(\mathbf{x})$ at a feasible point $\mathbf{x}$, and to

5   remove from consideration any subbox $\mathbf{X}$ for which

6          $f(\mathbf{X}) > f\_bar$;

7   wherein the term consistency mechanism is configured to apply term

8 consistency to the f_bar inequality $f(\mathbf{x}) \leq f\_bar$ over the subbox $\mathbf{X}$.

1   39.   The apparatus of claim 38, wherein the box consistency

2 mechanism is configured to apply box consistency to the f_bar inequality

3 $f(\mathbf{x}) \leq f\_bar$ over the subbox $\mathbf{X}$.

1   40.   The apparatus of claim 35, wherein if the subbox $\mathbf{X}$ is strictly

2 feasible ($p_i(\mathbf{X}) < 0$ for all $i=1,...,n$), the interval global optimization mechanism is

3 configured to:

4   determine a gradient $\mathbf{g}(\mathbf{x})$ of the function $f(\mathbf{x})$, wherein $\mathbf{g}(\mathbf{x})$ includes

5 components $g_i(\mathbf{x})$ ($i=1,...,n$);

40

1   remove from consideration any subbox for which $g(x)$ is bounded away

2   from zero, thereby indicating that the subbox does not include an extremum of

3    $f(x)$; and

4   the term consistency mechanism is configured to apply term consistency to

5   each component $g_i(x)=0$ ($i=1,...,n$) of $g(x)=0$ over the subbox X.

1   41.   The apparatus of claim 40, wherein the box consistency

2   mechanism is configured to apply box consistency to each component

3    $g_i(x)=0$ ($i=1,...,n$) of $g(x)=0$ over the subbox X.

1   42.   The apparatus of claim 35, wherein if the subbox X is strictly

2   feasible ($p_i(X) < 0$ for all $i=1,...,n$), the interval global optimization mechanism is

3   configured to:

4   determine diagonal elements $H_{ii}(x)$ ($i=1,...,n$) of the Hessian of the

5   function $f(x)$;

6   remove from consideration any subbox for which a diagonal element

7    $H_{ii}(X)$ of the Hessian over the subbox X is always negative, indicating that the

8   function f is not convex over the subbox X and consequently does not contain a

9   global minimum within the subbox X; and

10   the term consistency mechanism is configured to apply term consistency to

11   each inequality $H_{ii}(x) \geq 0$ ($i=1,...,n$) over the subbox X.

1   43.   The apparatus of claim 42, wherein the box consistency

2   mechanism is configured to apply box consistency to each inequality

3    $H_{ii}(x) \geq 0$ ($i=1,...,n$) over the subbox X.

41

1   44.   The apparatus of claim 35, wherein if the subbox $\mathbf{X}$ is strictly

2   feasible ($p_i(\mathbf{X}) < 0$ for all $i=1,...,n$), the interval global optimization mechanism is

3   configured to perform the interval Newton step by:

4   computing the Jacobian $\mathbf{J(x,X)}$ of the gradient of the function f evaluated

5   with respect to a point $\mathbf{x}$ over the subbox $\mathbf{X}$; and

6   computing an approximate inverse $\mathbf{B}$ of the center of $\mathbf{J(x,X)}$,

7   using the approximate inverse $\mathbf{B}$ to analytically determine the system $\mathbf{Bg(x)}$,

8   wherein $\mathbf{g(x)}$ is the gradient of the function $f(\mathbf{x})$, and wherein $\mathbf{g(x)}$ includes

9   components $g_i(\mathbf{x})$ ($i=1,...,n$).

1   45.   The apparatus of claim 44, the term consistency mechanism is

2   configured to apply term consistency to each component $(\mathbf{Bg(x)})_i = 0$ ($i=1,...,n$) to

3   solve for the variable x_i over the subbox $\mathbf{X}$.

1   46.   The apparatus of claim 44, the box consistency mechanism is

2   configured to apply box consistency to each component $(\mathbf{Bg(x)})_i = 0$ ($i=1,...,n$) to

3   solve for the variable x_i over the subbox $\mathbf{X}$.

1   47.   The apparatus of claim 35, wherein the interval Newton

2   mechanism is configured to perform the Newton step on the John conditions.

1   48.   The apparatus of claim 35,

2   wherein the interval global optimization mechanism is configured to:

3   linearize the set of inequality constraints to produce a set of

4   linear inequality constraints with interval coefficients that enclose

5   the nonlinear inequality constraints, and to

42

1   precondition the set of linear inequality constraints through

2   additive linear combinations to produce a set of preconditioned

3   linear inequality constraints; and

4   wherein the term consistency mechanism is configured to apply term

5 consistency to the set of preconditioned linear inequality constraints over the

6 subbox $\mathbf{X}$.

1   49.   The apparatus of claim 48, wherein the box consistency

2 mechanism is configured to apply box consistency to the set of preconditioned

3 linear inequality constraints over the subbox $\mathbf{X}$.

1   50.   The apparatus of claim 35, wherein the term consistency

2 mechanism is configured to:

3   symbolically manipulate an equation within the computer system to solve

4 for a term, $g(x'_j)$, thereby producing a modified equation $g(x'_j) = h(\mathbf{x})$, wherein

5 the term $g(x'_j)$ can be analytically inverted to produce an inverse function $g^{-1}(\mathbf{y})$;

6   substitute the subbox $\mathbf{X}$ into the modified equation to produce the equation

7 $g(X'_j) = h(\mathbf{X})$;

8   solve for $X'_j = g^{-1}(h(\mathbf{X}))$; and

9   intersect X'_j with the j-th element of the subbox $\mathbf{X}$ to produce a new

10 subbox $\mathbf{X}^+$;

11   wherein the new subbox $\mathbf{X}^+$ contains all solutions of the equation within

12 the subbox $\mathbf{X}$, and wherein the size of the new subbox $\mathbf{X}^+$ is less than or equal to

13 the size of the subbox $\mathbf{X}$.

1   51.   The apparatus of claim 35, wherein the interval Newton

2 mechanism is configured to:

1   compute $J(x,X)$, wherein $J(x,X)$ is the Jacobian of the function f evaluated
2   as a function of x over the subbox X; and to
3   determine if $J(x,X)$ is regular as a byproduct of solving for the subbox Y
4   that contains values of y that satisfy $M(x,X)(y-x) = r(x)$, where
5    $M(x,X) = BJ(x,X)$, $r(x) = -Bf(x)$, and B is an approximate inverse of the center of
6    $J(x,X)$.

44